1 Differentiability

In this section, we discuss the differentiability of functions.

Definition 1.1 (Differentiable function)**.** *Let $f(x)$ be a function. We say that f is* **differentiable** *at $x = a$ if*

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

exists. The value of this limit is called the derivative of f at $x = a$ and is denoted by $f'(a)$. We say that f is differentiable on (a, b) if f is differentiable at every point on (a, b).

The first property of differentiable function is that it must be continuous. Let's recall the definition of continuous function.

Definition 1.2 (Continuous function)**.** *Let $f(x)$ be a function. We say that f is* **continuous** *at $x = a$ if the limit of f at $x = a$ exists and*

$$\lim_{x\to a} f(x) = f(a).$$

We say that f is continuous on (a, b) if f is continuous at every point on (a, b).

Theorem 1.3. *If f is differentiable at $x = a$, then f is continuous at $x = a$.*

Proof. Suppose f is differentiable at $x = a$. Then

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{h\to 0} f(a+h) \\
&= \lim_{h\to 0}((f(a+h) - f(a)) + f(a)) \\
&= \lim_{h\to 0}\left(\left(\frac{(f(a+h) - f(a)}{h}\right) h + f(a)\right) \\
&= f'(a) \cdot 0 + f(a) \\
&= f(a)
\end{aligned}$$

Therefore f is continuous at $x = a$. □

However the converse of the above theorem is false. There exists function which is continuous at a point but not differentiable at that point. Here is an example.

Example 1.4. *Let*

$$f(x) = |x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Then

1. *f is continuous at $x = 0$.*
2. *f is not differentiable at $x = 0$.*

Proof. The graphs of $f(x) = |x|$ and its derivative are shown in Figure 1.

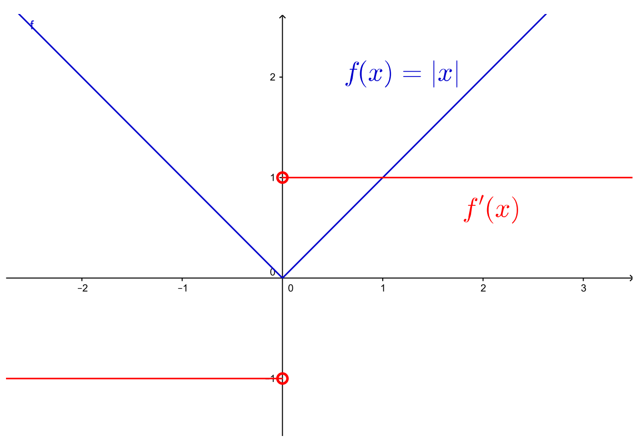

Figure 1: $f(x) = |x|$

1. The left and right-hand limits of $f(x) = |x|$ at $x = 0$ are

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} (-x) = 0 \\ \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} x = 0 \end{aligned}$$

Thus we have

$$\lim_{x \to 0} f(x) = 0 = f(0)$$

Therefore f is continuous at $x = 0$.

2. Observe that

$$\begin{aligned}\lim_{h\to 0^-}\frac{f(h)-f(0)}{h} &= \lim_{h\to 0^-}\frac{-h-0}{h}=-1\\ \lim_{h\to 0^+}\frac{f(h)-f(0)}{h} &= \lim_{h\to 0^+}\frac{h-0}{h}=1\end{aligned}$$

are not equal. Thus

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

does not exist. Therefore f is not differentiable at $x=0$.

□

Example 1.5. *Let*

$$f(x)=|x|\sin x=\begin{cases}-x\sin x, & \text{if } x<0\\ x\sin x, & \text{if } x\geq 0\end{cases}$$

Find $f'(x)$.

Solution. The graphs of $f(x)=|x|\sin x$ and its derivative are shown in Figure 2.
For $x<0$, we have

$$f'(x)=\frac{d}{dx}(-x\sin x)=-x\cos x-\sin x$$

For $x>0$, we have

$$f'(x)=\frac{d}{dx}(x\sin x)=x\cos x+\sin x$$

At $x=0$, we have

$$\begin{aligned}f'(0) &= \lim_{h\to 0}\frac{f(h)-f(0)}{h}\\ &= \lim_{h\to 0}\frac{|h|\sin h-0}{h}\\ &= \lim_{h\to 0}\left(\frac{\sin h}{h}\right)|h|\\ &= 0\end{aligned}$$

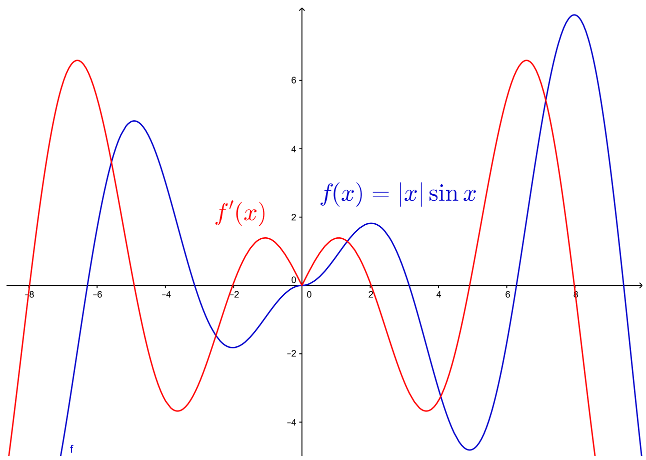

Figure 2: $f(x) = |x| \sin x$

Therefore

$$f'(x) = \begin{cases} -x\cos x - \sin x, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ x\cos x + \sin x, & \text{if } x > 0 \end{cases}$$

$\square$

Example 1.6. *Let*

$$f(x) = \begin{cases} x\sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Determine whether $f(x)$ is differentiable at $x = 0$.

Solution. The graphs of $f(x)$ and its derivative are shown in Figure 3. Since

$$\begin{aligned} \lim_{h\to 0} \frac{f(h) - f(0)}{h} &= \lim_{h\to 0} \frac{h\sin\left(\dfrac{1}{h}\right) - 0}{h} \\ &= \lim_{h\to 0} \sin\left(\dfrac{1}{h}\right) \end{aligned}$$

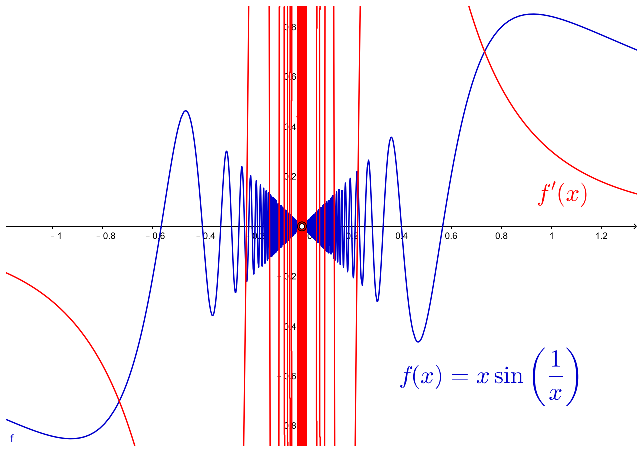

Figure 3: $f(x) = x\sin\left(\dfrac{1}{x}\right)$

does not exist. Therefore $f(x)$ is not differentiable at $x = 0$. $\square$

The following example is important because it shows that the derivative of a differentiable function can sometimes be discontinuous.

Example 1.7 (Function with discontinuous derivative)**.** *Let*

$$f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

1. *Find $f'(x)$ for $x \neq 0$.*
2. *Find $f'(0)$*
3. *Show that $f'(x)$ is not continuous at $x = 0$.*

Solution. The graphs of $f(x)$ and its derivative are shown in Figure 4.

1. When $x \neq 0$,

$$f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

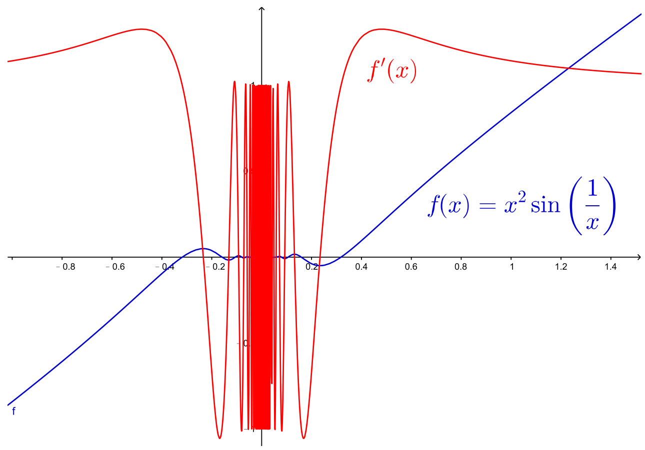

Figure 4: $f(x) = x^2 \sin\left(\dfrac{1}{x}\right)$

2.

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} \\ &= \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) \end{aligned}$$

Since $\left|\sin\left(\dfrac{1}{h}\right)\right| \leq 1$ is bounded and $\lim_{h\to 0} h = 0$, we have

$$f'(0) = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0$$

3. The limit

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} \left(2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right)$$

does not exist since $\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$ does not exist. Therefore $f'(x)$ is not continuous at $x = 0$. □

The differentiability at $x = 0$ of the functions in the above examples are summarized in the table below.

Example	$f(x)$	$f(x)$ is continuous at $x = 0$	$f(x)$ is differentiable at $x = 0$	$f'(x)$ is continuous at $x = 0$	Graph
1.4	$\|x\|$	Yes	No	Not applicable	Figure 1
1.5	$\|x\| \sin x$	Yes	Yes	Yes	Figure 2
1.6	$x \sin\left(\frac{1}{x}\right)$	Yes	No	Not applicable	Figure 3
1.7	$x^2 \sin\left(\frac{1}{x}\right)$	Yes	Yes	No	Figure 4

Note. In all of the examples above, we define $f(0) = 0$.

2 Mean Value Theorem

Imagine a vehicle traveling on a road. Suppose at time $t = a$ and $t = b$, the displacements of the vehicle are $f(a)$ and $f(b)$ respectively. Then the average velocity of the vehicle is

$$\frac{f(b) - f(a)}{b - a}$$

One may ask whether there always exists a time $t = \xi$ such that the velocity of the vehicle is exactly equal to the average velocity. Roughly speaking, the **mean value theorem** gives an affirmative answer to this question if we assume that velocity is defined at any time between a and b. The rigorous statement of mean value theorem is stated below.

Theorem 2.1 (Lagrange's mean value theorem)**.** *Let a, b be two real numbers with $a < b$. Suppose f is a function such that*

1. *f is continuous on $[a, b]$.*
2. *f is differentiable on (a, b).*

Then there exists $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

In the vehicle example above, $f(x)$ is the displacement of the vehicle and x is the time. Then $f'(x)$ is the velocity of the vehicle. One may wonder whether the following situation gives a counter example to the above theorem. Suppose from $t = 0$ to $t = 1$, the vehicle remains at rest and from $t = 1$ to $t = 2$, the vehicle travels with a velocity of 2 units. Then the average velocity of the vehicle is 1 but the velocity of vehicle is never equal to 1 from $t = 0$ to $t = 2$. This does not contradict the theorem because we assumed that f' is defined on (a, b) but velocity is not defined at $t = 1$.

Before we give the proof of the Lagrange's mean value theorem (Theorem 2.1), we state two variants of mean value theorem. The first one is a special case of the Lagrange's mean value theorem and the second one is a generalization of it.

Theorem 2.2 (Rolle's theorem)**.** *Let a, b be two real numbers with $a < b$. Suppose f is a function such that*

1. *f is continuous on $[a, b]$.*
2. *f is differentiable on (a, b).*
3. *$f(a) = f(b)$*

Then there exists $\xi \in (a, b)$ such that

$$f'(\xi) = 0$$

Theorem 2.3 (Cauchy's mean value theorem)**.** *Let a, b be two real numbers with $a < b$. Suppose f and g are functions such that*

1. *f and g are continuous on $[a, b]$.*
2. *f and g are differentiable on (a, b).*
3. *$g'(x) \neq 0$ for any $x \in (a, b)$*

Then there exists $\xi \in (a, b)$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Note that Rolle's theorem is a special case of Lagrange's mean value theorem. If we take $g(x) = x$ in the Cauchy's mean value theorem, we obtain Lagrange's mean value theorem. So Cauchy's mean value theorem is a generalization of Lagrange's mean value theorem. First we prove Rolle's theorem. The following theorem will be needed for this purpose.

Theorem 2.4 (Extreme value theorem)**.** *Suppose f is a function which is continuous on a closed and bounded interval $[a, b]$. Then there exists $p, q \in [a, b]$ such that*

$$f(p) \leq f(x) \leq f(q) \text{ for any } x \in [a, b].$$

In other words, f is bounded and attains both its maximum and minimum values.

The rigorous proof of the Extreme value theorem requires some argument in analysis and is omitted. We also need the following theorem whose proof is very easy.

Theorem 2.5. *Let a, b, ξ be real numbers such that $a < \xi < b$. Suppose f is differentiable at $x = \xi$ and either*

$$f(x) \leq f(\xi) \text{ for any } x \in (a, b),$$

or

$$f(x) \geq f(\xi) \text{ for any } x \in (a, b).$$

Then $f'(\xi) = 0$.

Proof. Suppose $f(x) \leq f(\xi)$ for any $x \in (a, b)$. The proof for the other case is more or less the same. For any $h < 0$ with $a < \xi + h < \xi$, we have

$$\frac{f(\xi + h) - f(\xi)}{h} \geq 0$$

Now $f'(\xi)$ exists and we have

$$f'(\xi) = \lim_{h \to 0^-} \frac{f(\xi + h) - f(\xi)}{h} \geq 0$$

On the other hand, for any $h > 0$ with $\xi < \xi + h < b$, we have

$$\frac{f(\xi + h) - f(\xi)}{h} \leq 0$$

Thus we also have

$$f'(\xi) = \lim_{h \to 0^+} \frac{f(\xi + h) - f(\xi)}{h} \leq 0$$

Therefore we have $f'(\xi) = 0$. □

Now we are ready to prove Rolle's theorem.

Proof of Rolle's theorem. Suppose f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$. By Extreme value theorem (Theorem 2.4) there exists $p, q \in [a, b]$ such that

$$f(p) \leq f(x) \leq f(q) \text{ for any } x \in [a, b].$$

If $p \in (a, b)$, i.e., $p \neq a, b$, then we take $\xi = p$. If $q \in (a, b)$, then we take $\xi = q$. If both p and q do not lie on (a, b), then f is a constant function and we take ξ to be any point in (a, b). In any of the above cases, we have $f'(\xi) = 0$ by Theorem 2.5. $\square$

Next we use Rolle's theorem to prove Lagrange's mean value theorem.

Proof of Lagrange's mean value theorem. Let

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a} x.$$

The function $g(x)$ is constructed so that

$$\begin{aligned} g(b) - g(a) &= \left(f(b) - \frac{f(b) - f(a)}{b - a} b \right) - \left(f(a) - \frac{f(b) - f(a)}{b - a} a \right) \\ &= (f(b) - f(a)) - \frac{f(b) - f(a)}{b - a}(b - a) \\ &= 0 \end{aligned}$$

Applying Rolle's theorem to $g(x)$ on $[a, b]$, there exists $\xi \in (a, b)$ such that $g'(\xi) = 0$ which means

$$f'(\xi) - \frac{f(b) - f(a)}{b - a} = 0$$

and the proof of Lagrange's mean value theorem is complete. $\square$

It is well known that a function with non-negative derivative is monotonic increasing. We may use Lagrange's mean value theorem to give a rigorous proof of this statement.

Theorem 2.6. *Let $f(x)$ be a function which is differentiable on (a, b). Suppose $f'(x) \geq 0$ for any $x \in (a, b)$. Then for any $x, y \in (a, b)$ with $x < y$, we have $f(x) \leq f(y)$.*

Proof. Suppose $f'(x) \geq 0$ for any $x \in (a, b)$ and $x, y \in (a, b)$ with $x < y$. By Lagrange's mean value theorem, there exists $\xi \in (x, y)$ such that

$$f(y) - f(x) = f'(\xi)(y - x)$$

which is non-negative since $f'(\xi) \geq 0$ and $y - x > 0$. This completes the proof of the theorem. $\square$

Proof of Cauchy's mean value theorem. Let f and g be functions which are continuous on $[a,b]$ and are differentiable on (a,b). Suppose $g'(x) \neq 0$ for any $x \in (a,b)$. First of all $g(a) \neq g(b)$, for otherwise $g'(\xi) = 0$ for some $\xi \in (a,b)$ by Rolle's theorem which violates our assumption on g. Thus we may let

$$h(x) = f(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g(x)$$

Then

$$\begin{aligned} h(b) - h(a) &= \left(f(b) - \frac{f(b)-f(a)}{g(b)-g(a)} g(b) \right) - \left(f(a) - \frac{f(b)-f(a)}{g(b)-g(a)} g(a) \right) \\ &= (f(b) - f(a)) - \frac{f(b)-f(a)}{g(b)-g(a)} (g(b) - g(a)) \\ &= 0 \end{aligned}$$

Applying Rolle's theorem to $h(x)$ on $[a,b]$, there exists $\xi \in (a,b)$ such that

$$\begin{aligned} h'(\xi) &= 0 \\ f'(\xi) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(\xi) &= 0 \\ f'(\xi) &= \frac{f(b)-f(a)}{g(b)-g(a)} g'(\xi) \\ \frac{f'(\xi)}{g'(\xi)} &= \frac{f(b)-f(a)}{g(b)-g(a)} \end{aligned}$$

Note that $g'(\xi) \neq 0$ since we assumed $g'(x) \neq 0$ for any x. This completes the proof of Cauchy's mean value theorem. □

3 L'Hopital Rule

In this section, we study an application of Cauchy's mean value theorem which gives a powerful tool to evaluate limits.

Theorem 3.1 (L'Hopital rule)**.** *Let $f(x)$ and $g(x)$ be functions and $a \in [-\infty, +\infty]$ (Here a can be $-\infty$ or $+\infty$) which satisfy*

1. *$f(x)$ and $g(x)$ are differentiable for any $x \neq a$.*
2. *$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, (or $\lim_{x\to a} f(x), \lim_{x\to a} g(x) = \pm\infty$).*
3. *$g'(x) \neq 0$ for any $x \neq a$.*
4. *$\lim_{x\to a} \frac{f'(x)}{g'(x)} = l$*

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = l$$

Proof. For simplicity, we assume $a \neq \pm\infty$ and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. Redefine f and g, if necessary, so that $f(a) = g(a) = 0$. Then for any $x > a$, we have

1. f and g are continuous on $[a, x]$.
2. f and g are differentiable on (a, x).
3. $g'(y) \neq 0$ for any $y \in (a, x)$.

By Cauchy's mean value theorem (Theorem 2.3), there exists $\xi \in (a, x)$, (here ξ depends on x), such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x)}{g(x)}$$

Therefore

$$\begin{aligned} \lim_{x\to a^+} \frac{f(x)}{g(x)} &= \lim_{x\to a^+} \frac{f'(\xi)}{g'(\xi)} \\ &= \lim_{\xi\to a^+} \frac{f'(\xi)}{g'(\xi)} \\ &= l \end{aligned}$$

Note that $\xi \to a$ as $x \to a$ since $\xi \in (a, x)$. The same is true for the left-hand limit and the proof of L'Hopital rule is complete. $\square$

Let's show how L'Hopital rule still works when $a = +\infty$. Define $F(y) = f(\frac{1}{y})$ and $G(y) = g(\frac{1}{y})$. Then

$$F'(y) = -\frac{f'(\frac{1}{y})}{y^2} \text{ and } G'(x) = -\frac{g'(\frac{1}{y})}{y^2}$$

Applying L'Hopital rule to $F(x)$ and $G(x)$ at $x = 0$, we have

$$\begin{aligned} \lim_{x\to+\infty} \frac{f(x)}{g(x)} &= \lim_{y\to 0^+} \frac{F(y)}{G(y)} \\ &= \lim_{y\to 0^+} \frac{F'(y)}{G'(y)} \\ &= \lim_{y\to 0^+} \frac{-\frac{f'(\frac{1}{y})}{y^2}}{-\frac{g'(\frac{1}{y})}{y^2}} \\ &= \lim_{y\to 0^+} \frac{f'(\frac{1}{y})}{g'(\frac{1}{y})} \\ &= \lim_{x\to+\infty} \frac{f'(x)}{g'(x)} \end{aligned}$$

Other cases of L'Hopital rule can be proved in similar ways.

Example 3.2. *Evaluate the following limits.*

1. $\displaystyle\lim_{x\to 0} \frac{e^{3x} - e^{-x}}{\sin x}$
2. $\displaystyle\lim_{x\to 0} \frac{e^{3x^2} - 1}{\cos x - \cos 2x}$
3. $\displaystyle\lim_{x\to 0^+} \frac{\ln(1 - \cos x)}{\ln \sin x}$
4. $\displaystyle\lim_{x\to+\infty} \frac{\ln(5x^3 - 2x + 3)}{\ln(4x^2 + 1)}$

Solution.

1. Since
$$\lim_{x\to 0}(e^{3x}-e^{-x}) = \lim_{x\to 0}\sin x = 0,$$
we may apply L'Hopital rule and get
$$\begin{aligned}\lim_{x\to 0}\frac{e^{3x}-e^{-x}}{\sin x} &= \lim_{x\to 0}\frac{3e^{3x}+e^{-x}}{\cos x}\\ &= \frac{3+1}{1}\\ &= 4\end{aligned}$$

2. Applying L'Hopital rule two times, we get
$$\begin{aligned}\lim_{x\to 0}\frac{e^{3x^2}-1}{\cos x-\cos 2x} &= \lim_{x\to 0}\frac{6xe^{3x^2}}{-\sin x+2\sin 2x}\\ &= \lim_{x\to 0}\frac{36x^2e^{3x^2}+6e^{3x^2}}{-\cos x+4\cos 2x}\\ &= \frac{6}{-1+4}\\ &= 2\end{aligned}$$

3. Since
$$\lim_{x\to 0^+}\ln(1-\cos x) = \lim_{x\to 0^+}\ln\sin x = -\infty,$$
we may apply L'Hopital rule and get
$$\begin{aligned}\lim_{x\to 0^+}\frac{\ln(1-\cos x)}{\ln\sin x} &= \lim_{x\to 0^+}\frac{\frac{\sin x}{1-\cos x}}{\frac{\cos x}{\sin x}}\\ &= \lim_{x\to 0^+}\frac{\sin^2 x}{(1-\cos x)\cos x}\\ &= \lim_{x\to 0^+}\frac{1-\cos^2 x}{(1-\cos x)\cos x}\\ &= \lim_{x\to 0^+}\frac{1+\cos x}{\cos x}\\ &= 2\end{aligned}$$

4. Since
$$\lim_{x\to +\infty}\ln(5x^3-2x+3) = \lim_{x\to +\infty}\ln(4x^2+1) = +\infty$$

we may apply L'Hopital rule and get

$$\begin{aligned}
\lim_{x\to+\infty}\frac{\ln(5x^3-2x+3)}{\ln(4x^2+1)} &= \lim_{x\to+\infty}\frac{\frac{15x^2-2}{5x^3-2x+3}}{\frac{8x}{4x^2+1}} \\
&= \lim_{x\to+\infty}\frac{(4x^2+1)(15x^2-2)}{8x(5x^3-2x+3)} \\
&= \lim_{x\to+\infty}\frac{(4+\frac{1}{x^2})(15-\frac{2}{x^2})}{8(5-\frac{2}{x^2}+\frac{3}{x^3})} \\
&= \frac{3}{2}
\end{aligned}$$

□

The limits in the above examples are of the forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. L'Hopital rule can also be used to evaluate limits of the forms $0\cdot\infty$, 0^0, ∞^0 and 1^∞.

Example 3.3. *Evaluate the following limits.*

1. $\lim_{x\to0^+} x\ln x$

2. $\lim_{x\to0^+} x^x$

3. $\lim_{x\to+\infty}(x^2+1)^{\frac{1}{\ln x}}$

4. $\lim_{x\to0}(\cos x)^{\frac{1}{x^2}}$

Solution.

1.

$$\begin{aligned}
\lim_{x\to0^+} x\ln x &= \lim_{x\to0^+}\frac{\ln x}{\frac{1}{x}} \\
&= \lim_{x\to0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} \\
&= \lim_{x\to0^+}(-x) \\
&= 0
\end{aligned}$$

2.

$$\begin{aligned}\ln\left(\lim_{x\to 0^+} x^x\right) &= \lim_{x\to 0^+} \ln(x^x) \\ &= \lim_{x\to 0^+} x\ln x \\ &= 0\end{aligned}$$

Therefore

$$\lim_{x\to 0^+} x^x = e^0 = 1$$

3.

$$\begin{aligned}\ln\left(\lim_{x\to +\infty} (x^2+1)^{\frac{1}{\ln x}}\right) &= \lim_{x\to +\infty} \ln((x^2+1)^{\frac{1}{\ln x}}) \\ &= \lim_{x\to +\infty} \frac{\ln(x^2+1)}{\ln x} \\ &= \lim_{x\to +\infty} \frac{\frac{2x}{x^2+1}}{\frac{1}{x}} \\ &= \lim_{x\to +\infty} \frac{2x^2}{x^2+1} \\ &= 2\end{aligned}$$

Therefore

$$\lim_{x\to +\infty} \ln((x^2+1)^{\frac{1}{\ln x}}) = e^2$$

4.

$$\begin{aligned}\ln\left(\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}}\right) &= \lim_{x\to 0} \ln((\cos x)^{\frac{1}{x^2}}) \\ &= \lim_{x\to 0} \frac{\ln\cos x}{x^2} \\ &= \lim_{x\to 0} \frac{\tan x}{2x} \\ &= \lim_{x\to 0} \frac{\sec^2 x}{2} \\ &= \frac{1}{2}\end{aligned}$$

Therefore

$$\lim_{x\to 0} (\cos x)^{\frac{1}{x^2}} = e^{\frac{1}{2}}$$